

# Entanglement in the scattering process by local impurity

Dong Yang

*Department of Modern Physics, University of Science and Technology of China,  
Hefei, Anhui 230026, People's Republic of China and  
Zhejiang Institute of Modern Physics, Zhejiang University,  
Hangzhou, Zhejiang 310027, People's Republic of China*

Shi-Jian Gu

*Department of Physics, The Chinese University of Hong Kong, Hong Kong, China and  
Zhejiang Institute of Modern Physics, Zhejiang University,  
Hangzhou, Zhejiang 310027, People's Republic of China*

Haibin Li

*Zhejiang Institute of Modern Physics, Zhejiang University,  
Hangzhou, Zhejiang 310027, People's Republic of China*

(Dated: February 1, 2008)

## Abstract

We study entanglement in the scattering processes by fixed impurity and Kondo impurity. The fixed impurity plays a role as spin state filter that is employed to concentrate entanglement between the scattering particle and the unscattering particle. One Kondo impurity can entangle two noninteracting scattering particles while one scattering particle can entangle two separate noninteracting Kondo impurities.

PACS numbers: 03.67.-a

## I. INTRODUCTION

Entanglement lies at the heart of quantum information and quantum computation [1]. It is responsible for most of quantum phenomena, such as quantum teleportation, dense coding, quantum cryptography [2]. Now it is regarded as a kind of useful resource in quantum information process. Much efforts were devoted to the entanglement capacity of unitary evolution and interaction Hamiltonian. For example, Childs et al. [4] found an explicit formula for the maximum entanglement created by a class of two-qubit Hamiltonians, including the Ising interaction and the anisotropic Heisenberg interaction. Nielsen et al [3] developed a theory quantifying the strength of quantum dynamical operations. Experimentally, entanglement creation and distillation are of importance [5]. Currently, these tasks are mainly realized by optics. In this paper, We study the scattering process by the  $\delta$  interaction created by the fixed impurity and Kondo impurity and show that entanglement distillation and creation can be realized in the scattering process.

The paper is structured as follows. Section II discusses the scattering process with fixed impurity and introduces spin state filter that is later employed to distill entanglement. Section III considers the scattering process with Kondo impurity that is utilized to create entanglement. Section IV concludes with a summary.

## II. SPIN STATE FILTER

In this section, we discussed the scattering process that a free particle with spin is scattered by an impurity with fixed spin. We show that the scattering process plays a role as spin state filter, then we utilize the filter to concentrate entanglement between two non-maximally entangled particles.

First let us consider the problem of the free particle scattering with the  $\delta$  potential. The Hamiltonian is

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + r\delta(x), \quad (1)$$

where we set  $\hbar = m = 1$  and  $r$  is the strength of the  $\delta$  potential. The  $\delta$  potential comes from a local impurity without a spin. Suppose the incident wave function is  $\phi_i = e^{ikx}$ , then

the solution of the scattering process is

$$\phi_f = \begin{cases} e^{ikx} + Re^{-ikx}, & x < 0, \\ Se^{ikx}, & x > 0, \end{cases} \quad (2)$$

where  $S = 1/(1 + i\xi)$  is the transmission amplitude,  $R = S - 1$  the refraction amplitude,  $\xi = r/k$ .

Now suppose a free particle with spin is scattered by a impurity with fixed spin state. The situation occurs where an free electron is scattered by a localized electron whose spin is fixed, say spin up. If the electric interaction is neglected, the scattering process can be described by the Toy model,

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + r\delta(x)(1 - \sigma_z), \quad (3)$$

where  $\sigma_z$  is the Pauli operator. In the eigenbasis of  $\sigma_z$ , the potential is written as  $2r\delta(x) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ . That is to say the spin of the particle is visible to the impurity on the origin. If the particle is spin-up, the impurity lets the particle go through freely, just as the impurity is nonexisting. If the particle is spin-down, the impurity acts as the  $2r\delta(x)$  potential. For the general case, where the particle lies in the coherent state of spin-up and spin-down, The problem can be solved in the spirit of fraction wave method: the potential is diagonalised and the eigenvalues and eigenstates are found; for each of the eigenstates, the scattering problem is solved for the potential determined by its corresponding eigenvalue; the incident wave state is expanded in the eigenstates and the scattered wave state is the superposition of the scattered wave states of the eigenstates. Now suppose the incident wave state is expanded as

$$\phi_i = e^{ikx}(\alpha|0\rangle + \beta|1\rangle), \quad (4)$$

where  $|0\rangle$  is spin-up and  $|1\rangle$  spin-down and  $|\alpha|^2 + |\beta|^2 = 1$ . The solution can be obtained,

$$\phi_f = \begin{cases} \alpha e^{ikx}|0\rangle + \beta(e^{ikx} + Re^{-ikx})|1\rangle, & x < 0, \\ e^{ikx}(\alpha|0\rangle + \beta S|1\rangle), & x > 0 \end{cases} \quad (5)$$

where  $S = 1/(1 + i\xi)$ ,  $\xi = 2r/k$ . Note that in the transmitted part, the amplitude of the spin-down is decreased. The effect of the scattering process on the spin state is like a spin filter.

Spin filter can be employed to distill entanglement between two non-maximally entangled particles [6]. Suppose the state of two particles is

$$\phi_i = e^{-ikx_2} e^{ikx_1} (a|00\rangle + b|11\rangle), a < b, \quad (6)$$

in which  $|a|^2 + |b|^2 = 1$ . The entanglement of pure bipartite state is measured by von Neumann entropy of the reduced state of any particle,  $E = -\text{tr} \rho_1 \log \rho_1 = -|a|^2 \log |a|^2 - |b|^2 \log |b|^2$ , where  $\rho_1 = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$ . Suppose particle one is scattered by the impurity. According to the fraction wave method,

$$\begin{aligned} e^{ikx_1} |0\rangle &\rightarrow \begin{cases} e^{ikx_1} |0\rangle, & x < 0, \\ e^{ikx_1} |0\rangle, & x > 0, \end{cases} \\ e^{ikx_1} |1\rangle &\rightarrow \begin{cases} (e^{ikx_1} + Re^{-ikx_1}) |1\rangle, & x < 0, \\ Se^{ikx_1} |1\rangle, & x > 0. \end{cases} \end{aligned} \quad (7)$$

After the scattering process, the two particle state is of the form,

$$\phi_f = \begin{cases} ae^{-ikx_2} e^{ikx_1} |00\rangle + be^{-ikx_2} (e^{ikx_1} + Re^{-ikx_1}) |11\rangle, & x < 0, \\ e^{-ikx_2} e^{ikx_1} (a|00\rangle + bS|11\rangle), & x > 0. \end{cases} \quad (8)$$

We concern mainly about the transmitted part. When  $|a| = |bS|$ , the transmitted state is maximally entangled. Given  $a, k$ , we can modify the strength  $r$  to satisfy  $|a| = |bS|$ . Actually, we can vary the direction of the impurity to achieve the same effect in a simpler way. Notice that the probability to obtain a maximally entangled state is less than the maximum one  $2|a|^2$  that can be obtained theoretically. The reason lies at the refracted part still contains entanglement as the refracted term is not zero. Indeed, there exists the optimal direction of the impurity that the probability to obtain maximally entangled state is maximum by the scattering process.

### III. CREATING ENTANGLEMENT

In this section, we introduce the scattering process that a free particle are scattered by Kondo impurity. Then we show how to utilize the process to create entanglement between two noninteracting particles and how to create entanglement between two noninteracting

Kondo impurities. A complementary relation is given between the strength of the  $\delta$  potential and the rate of entanglement creation.

The impurity is called Kondo impurity if the spin of the impurity is free and interacts with the particle. The scattering process is described by the Kondo model:

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + r\delta(x)\vec{\sigma}_1 \cdot \vec{\sigma}_0, \quad (9)$$

where  $\vec{\sigma}_1$  is the spin vector operator of the particle and  $\vec{\sigma}_0$  that of the Kondo impurity. The eigenbasis of  $\vec{\sigma}_1 \cdot \vec{\sigma}_0$  is,

$$\begin{aligned} |\lambda_1\rangle &= |00\rangle, \\ |\lambda_2\rangle &= |11\rangle, \\ |\lambda_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\lambda_4\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \end{aligned} \quad (10)$$

where  $\lambda_i$  are the corresponding eigenvalues,  $\lambda_1 = \lambda_2 = 1, \lambda_3 = -2, \lambda_4 = 0$ . In the eigenbasis,  $H$  is expressed as

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + r\delta(x) \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| \quad (11)$$

Solving the scattering process when the incident wave state is  $\phi_i = e^{ikx}|\lambda_i\rangle$  gives,

$$\phi_f = \begin{cases} (e^{ikx} + R_i e^{-ikx})|\lambda_i\rangle, & x < 0, \\ S_i e^{ikx}|\lambda_i\rangle, & x > 0, \end{cases} \quad (12)$$

where  $S_i = 1/(1 + i\xi_i)$ ,  $\xi_i = r\lambda_i/k$ .

Generally the incident wave state is

$$\phi_i = e^{ikx}|\chi\rangle = e^{ikx} \sum_i c_i |\lambda_i\rangle, \quad (13)$$

and the scattering state is

$$\phi_f = \begin{cases} \sum_i c_i (e^{ikx} + R_i e^{-ikx})|\lambda_i\rangle, & x < 0, \\ e^{ikx} \sum_i c_i S_i |\lambda_i\rangle, & x > 0, \end{cases} \quad (14)$$

After the scattering process  $\phi_f$  is projected onto two subspaces: one is  $x < 0$ , and the other is  $x > 0$ . The transmitted part  $x > 0$  is retained while that  $x < 0$  is discarded. In other words, we only concern about the scattering term. Notice that in the scattering term

the wave function of coordinate is the same as the incident wave. Because we pay attention to entanglement between the spins, we omit the wave function of coordinate and rewrite the effect on the spins as following,

$$\begin{aligned}
|00\rangle &\rightarrow S_1|00\rangle, \\
|11\rangle &\rightarrow S_2|11\rangle, \\
|01\rangle &\rightarrow \frac{S_3 + S_4}{2}|01\rangle + \frac{S_3 - S_4}{2}|10\rangle, \\
|10\rangle &\rightarrow \frac{S_3 - S_4}{2}|01\rangle - \frac{S_3 + S_4}{2}|10\rangle.
\end{aligned} \tag{15}$$

Suppose initially the impurity is polarized in spin-up and the particle is in a general state,

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \rightarrow \alpha S_1|00\rangle + \frac{\beta(S_3 - S_4)}{2}|01\rangle - \frac{\beta(S_3 + S_4)}{2}|10\rangle. \tag{16}$$

Once the particle is transmitted, the spin state of the particle is entangled with that of the Kondo impurity. It appears that the Kondo impurity does not play a role as spin state filter. However, Kondo impurity combining with additional measurement on it would have the same effect as spin state filter. Here we use it to concentrate entanglement as the fixed impurity. Of course, extra measurement on the Kondo impurity is required after scattering. As the case discussed in the fixed impurity,

$$(a|00\rangle + b|11\rangle)|0\rangle \rightarrow \left( aS_1|00\rangle - \frac{b(S_3 + S_4)}{2}|11\rangle \right) |0\rangle + \frac{b(S_3 - S_4)}{2}|10\rangle|1\rangle. \tag{17}$$

If  $|aS_1| = |\frac{b(S_3 + S_4)}{2}|$ , then the projection onto  $|0\rangle\langle 0|$  will give the maximally entangled state. Notice that different measurements can be performed on the Kondo impurity. The capacity of the concentrating entanglement of the two kinds of impurities can be compared.

An important property of Kondo impurity is that it can be employed to create entanglement between two noninteracting scattering particles. The process is as following. Suppose initially the Kondo impurity  $P_0$  is polarized in  $|1\rangle_0$ , the two noninteracting particles in product state  $|0\rangle_2|0\rangle_1$ . First  $P_1$  is scattered by the Kondo impurity. If  $P_1$  is transmitted, then  $P_2$  is scattered by the impurity. If  $P_1$  is refracted, the impurity  $P_0$  is polarized again in  $|1\rangle_0$  and a new  $P_1$  is incident on the impurity. In all, we just consider the possibility when  $P_1$  and  $P_2$  are scattered with the same impurity  $P_0$  one by one and both particles are transmitted.

The evolution is

$$\begin{aligned}
& |0\rangle_2|0\rangle_1|1\rangle_0 \rightarrow |0\rangle_2 \left( \frac{S_3^1 + S_4^1}{2} |0\rangle_1|1\rangle_0 + \frac{S_3^1 - S_4^1}{2} |1\rangle_1|0\rangle_0 \right) \\
& \rightarrow \frac{S_3^2 + S_4^2}{2} \frac{S_3^1 + S_4^1}{2} |0\rangle_2|0\rangle_1|1\rangle_0 + \left( \frac{S_3^2 - S_4^2}{2} \frac{S_3^1 + S_4^1}{2} |1\rangle_2|0\rangle_1 + \frac{S_1^2(S_3^1 - S_4^1)}{2} |0\rangle_2|1\rangle_1 \right) |0\rangle_0, \quad (18)
\end{aligned}$$

where the first arrow denotes the  $P_1$  is scattered and the second arrow denotes the resulted transmitted state when both particles are transmitted. Measurement on the impurity is performed in the basis  $\{|0\rangle, |1\rangle\}$ . Once the outcome  $|0\rangle$  occurs,  $P_1$  is entangled with  $P_2$ . Here we remark that the initial spin polarized directions are of importance. Explicitly, when the initial state of the three particles is  $|0\rangle_2|0\rangle_1|0\rangle_0$ , no entanglement will exist in the final transmitted term.

Exchanging the roles of the scattering particle and the impurity, a moving particle can be used to create entanglement between two separate noninteracting impurities. Suppose  $P_1, P_2$  are two Kondo impurities located at  $-a$  and  $a$  respectively. A particle  $P_0$  moves from left to right scattering with  $P_1$  and  $P_2$  sequently. The evolution is described by the Hamiltonian,

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + r_1 \delta(x+a) \vec{\sigma} \cdot \vec{\sigma}_1 + r_2 \delta(x-a) \vec{\sigma} \cdot \vec{\sigma}_2. \quad (19)$$

Strictly speaking, we should solve the multipartite scattering problem. Here we mainly consider the first order scattering that means the contributions that refract to and fro between the impurities is neglected. The transmitted part  $x > a$  is mainly from the direct transmission. Suppose the initial state is  $|1\rangle_0|0\rangle_1|0\rangle_2$ , the evolution is

$$\begin{aligned}
& |1\rangle_0|0\rangle_1|0\rangle_2 \rightarrow \left( \frac{S_3^1 - S_4^1}{2} |0\rangle_0|1\rangle_1 - \frac{S_3^1 + S_4^1}{2} |1\rangle_0|0\rangle_1 \right) |0\rangle_2 \\
& \rightarrow \frac{S_3^1 + S_4^1}{2} \frac{S_3^2 + S_4^2}{2} |1\rangle_0|0\rangle_1|0\rangle_2 + |0\rangle_0 \left( \frac{S_1^2(S_3^1 - S_4^1)}{2} |1\rangle_1|0\rangle_2 - \frac{S_3^1 + S_4^1}{2} \frac{S_3^2 - S_4^2}{2} |0\rangle_1|1\rangle_2 \right) \quad (20)
\end{aligned}$$

Measurement on the moving particle is performed in the basis  $\{|0\rangle, |1\rangle\}$ . The occurrence of the outcome  $|0\rangle$  will reduce the two impurities to entangled state.

#### IV. SUMMARY

In this article, we showed the scattering process can be employed to tackle with entanglement problem. Spin state filter can be realized by scattering with fixed impurity. Entanglement between two noninteracting particles can be created by scattering with the Kondo

impurity. On the contrary, entanglement between two noninteracting Kondo impurities can be created by scattering with the same particle. In the case of entanglement creation, we just considered the scattering process with the first order. Strictly speaking, the scattering process with high orders should be included and deserves further investigation.

- 
- [1] M. A. Nilesen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000)
  - [2] C. H. Bennett and G. Brassard, In Proceedings of IEEE International Conference on Computers, Sustems and Signal Processing, Pages 175-179, New York, 1984; C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett., **68**, 557 (1992); C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. Wootters, Phys. Rev. Lett., **70**, 1895 (1993).
  - [3] M. A. Nielsen, C. M. Dawson, J. L. Dodd, A. Gilchrist, D. Mortimer, T. J. Osborne, M. J. Bremner, A. W. Harrow, and A. Hines, Phys. Rev. A **67**, 052301 (2003).
  - [4] A. M. Childs, D. W. Leung, F. Verstraete, and G. Vidal, Quantum Inf. Comput. **3**, 97 (2003).
  - [5] C. H. Bennett, D.P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
  - [6] N. Gisin, Phys. Lett. A **210**, 151 (1996).